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# 2-STRONG PRODUCT AND CONNECTEDNESS

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**Abstract:** Among all the product of two graphs $G$ and $H$, the Cartesian product $G \times H$, tensor product $G \otimes H$ and strong product $G \boxtimes H$ are very well known and studied in detail. Recently, Cartesian product and tensor product were generalized by defining 2-Cartesian product and 2-tensor product and their properties were studied. In this paper, we have generalized strong product of two graphs G and H by defining 2-strong product $G \boxtimes_2 H$ and studied some basic graph parameters like connectedness and distance.

**Keywords and Phrases:** Strong product of graphs, connected graph, distance, diameter, eccentricity, radius.

**2020 Mathematics Subject Classification:** 05C07, 05C12.

## 1. Introduction and Preliminaries

Product of two graphs $G$ and $H$ has been defined in many different ways in literature. Among all the products, Cartesian product $G \times H$, tensor product

$G \otimes H$, strong product $G \boxtimes H$ and lexicographic product $G \circ H$ are studied in detail ([5] & [6]). Cartesian product and tensor product have been generalized by defining 2-Cartesian product $G \times_2 H$ and 2-tensor product $G \otimes_2 H$ in [2] & [3] and basic parameters of both the graphs have been obtained in terms of the parameters of $G$ and $H$.

In this paper, we generalize strong product of two graphs $G$ and $H$ by defining 2-strong product $G \boxtimes_2 H$ and study some basic graph parameters.

For any graph to be called connected, every vertex pair in it should be connected by a path. Distance between two vertices $x$ and $x'$ in graph $G$ is defined as the length of shortest path between $x$ and $x'$. For rest of the basic definitions in graph, we refer [4].

**Definition 1.1.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two connected graphs. Then 2-strong product of $G$ and $H$ is a graph denoted by $G \boxtimes_2 H$ with vertex set as $V(G \boxtimes_2 H) = V(G) \times V(H)$ and edge set as*
*$E(G \boxtimes_2 H) = \{(x, y)(x', y') : (x = x' \;\&\; d_H(y, y') = 2) \text{ or } (d_G(x, x') = 2 \;\&\; y = y')$*
*or $(d_G(x, x') = 2 \,\&\, d_H(y, y') = 2)\}$. Equivalently, $d_G(x, x') \in \{0, 2\}$ and*
*$d_H(y, y') \in \{0, 2\}$.*

If we replace 2 by 1, we get the definition of usual strong product $G \boxtimes H$.

**Remark 1.2.**

1. *$|E(G \boxtimes_2 H)| = |V(G)||N^2(H)| + |V(H)||N^2(G)| + 2|N^2(G)||N^2(H)|$,*
   *where $N^2(G)$ denotes the collection of all distinct pair of vertices which are at distance 2. Hence, we always assume that $N^2(G)$ and $N^2(H)$ are non-empty sets. Otherwise, we get $G \boxtimes_2 H$ as null graph.*

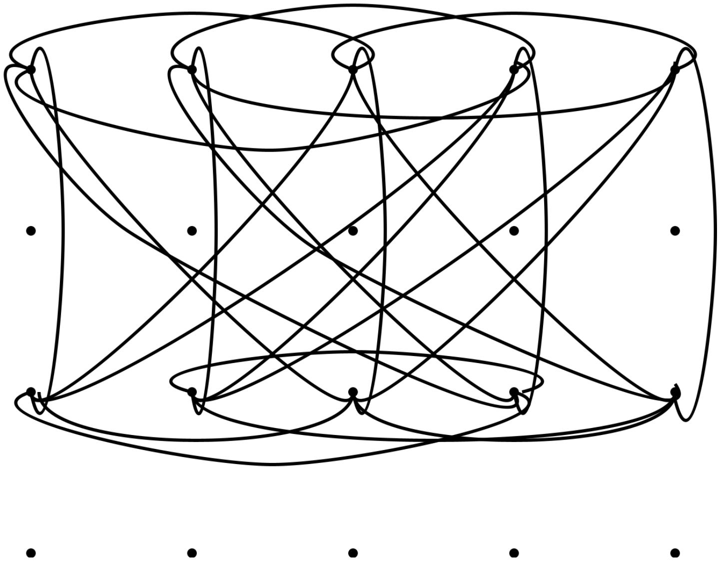
2. *$G \boxtimes_2 H \cong H \boxtimes_2 G$.*

3. *$G \boxtimes_2 H = (G \times_2 H) \cup (G \otimes_2 H)$, where $G \times_2 H$ is the 2-Cartesian product of $G$ and $H$, $G \otimes_2 H$ is the 2-tensor product of $G$ and $H$ and $\cup$ denotes the edge disjoint union of $G \times_2 H$ and $G \otimes_2 H$ ([2], [3]).*

**Example 1.3.**

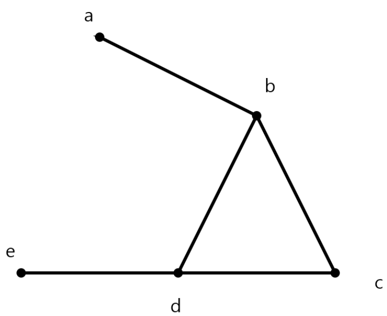
1. $P_4 \boxtimes_2 C_4$ has four components as $K_4$.

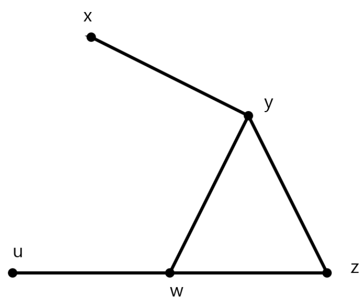
2. One component of 2-strong product of two graphs $P_4$ and $C_5$, $P_4 \boxtimes_2 C_5$ is shown below.



3. Let $G$ & $H$ be the graphs shown below.

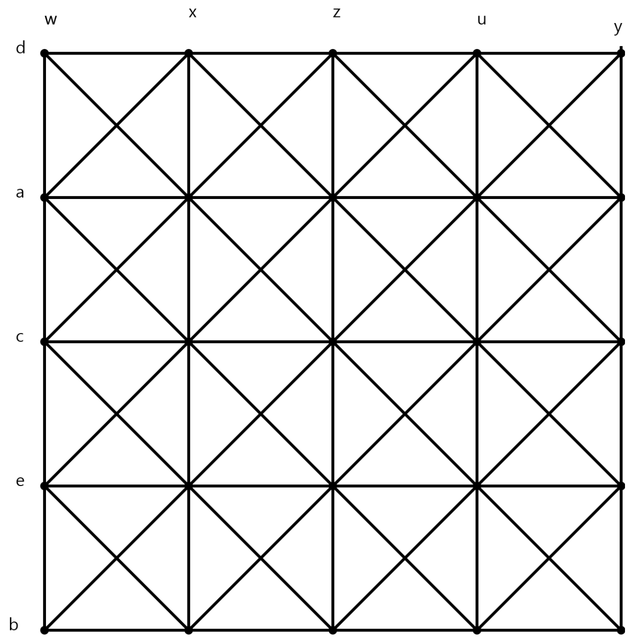
Figure 1: $G$ $H$




Then, $G \boxtimes_2 H$ is shown below.

Figure 2: $G \boxtimes_2 H$




First, we obtain basic graph parameters of $G \boxtimes_2 H$ such as degree of vertex, regularity and Eulerian property.

**Proposition 1.4.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. Then, for $(x, y) \in V(G \boxtimes_2 H)$,*

$$\deg_{G\boxtimes_2 H}((x, y)) = \deg_2(x) + \deg_2(y) + \deg_2(x)\deg_2(y),$$

*where* $\deg_2(x) = |N^2(x)|$ *and* $N^2(x) = \{x' \in V(G) : d_G(x, x') = 2\}$.

**Proof.** Let $(x, y) \in V(G \boxtimes_2 H)$. Then all vertices $(x, y')$, where $y' \in V(H)$ with $d_H(y, y') = 2$ are adjacent to $(x, y)$. Note that there are $deg_2(y)$ such vertices. Similarly, all vertices $(x', y)$, where $x' \in V(G)$ with $d_G(x, x') = 2$ are adjacent to $(x, y)$. There are $deg_2(x)$ such vertices. Also, note that $(x', y') \in V(G \boxtimes_2 H)$ where $d_G(x, x') = 2$ and $d_H(y, y') = 2$ is adjacent to $(x, y)$ and there are $deg_2(x)deg_2(y)$ such vertices. Thus, degree of $(x, y)$ in $G \boxtimes_2 H$ is
$deg_2(x) + deg_2(y) + deg_2(x)deg_2(y)$.

Now, we obtain a sufficient condition of regularity in $G\boxtimes_2 H$, using the following definition of second regular graph given in [7].

**Definition [7].** *A graph $G = (V(G), E(G))$ is said to be second regular with* *regularity $k$ if $deg_2(x) = k$, for all $x \in V(G)$.*

**Corollary 1.5.** *If graphs $G$ and $H$ are second regular graphs with regularity $k_1$ and $k_2$ respectively, then $G \boxtimes_2 H$ is $k_1 + k_2 + k_1k_2$ regular graph.*

In general, if $G$ and $H$ are Euler graphs, then $G \boxtimes_2 H$ may not be an Euler graph.

**Example 1.6.** Let $G = H$ be the graph shown below.

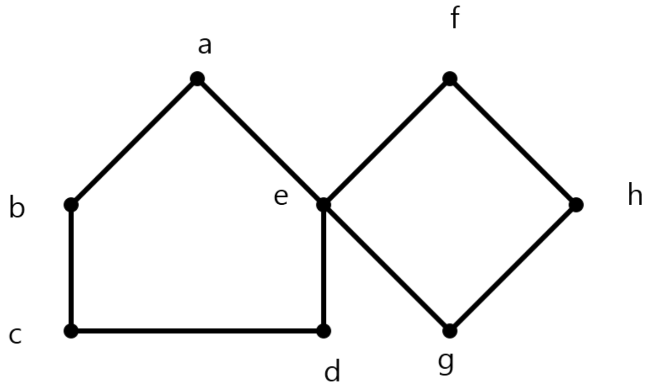



Figure 3: $G = H$

Then, $G$ and $H$ are Euler graphs, but $G \boxtimes_2 H$ is not Euler, as $deg_{G\boxtimes_2 H}(e, e)$ is odd, using Proposition 1.4.

Now, we give a characterization for $G \boxtimes_2 H$ to be Euler graph, in terms of 2 degree of a vertex.

**Theorem 1.7.** *Let $G = (V(G), V(G))$ and $H = (V(H), E(H))$ be two connected graphs, such that $G \boxtimes_2 H$ is connected. Then, $deg_2(x)$ is even for every $x \in V(G)$ and $deg_2(y)$ is also even for every $y \in V(H)$ if and only if $G \boxtimes_2 H$ is an Euler graph.*

**Proof.** Follows from Proposition 1.4.

**Remark 1.8.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs with $N^2(x) \neq \emptyset$ for some $x \in V(G)$ and $N^2(y) \neq \emptyset$ for some $y \in V(H)$. Then $K_4$ is a subgraph of $G \boxtimes_2 H$ and hence $G \boxtimes_2 H$ is a non-bipartite graph.*

## 2. Distance and Connectedness

In this section, we obtain the distance formula and discuss connectedness of $G \boxtimes_2 H$.

In usual strong product $G \boxtimes H$ of two graphs $G$ and $H$, distance formula is given as follows:

**Theorem.** [5] *For two vertices $(x, y)$ and $(x', y')$ in $G \boxtimes H$,*

$$d_{G\boxtimes H}((x,y),(x',y')) = \max\{d_G(x,x'), d_H(y,y')\}.$$

To obtain distance formula for $G \boxtimes_2 H$, we shall need the following definition.

**Definition.** [7] *Let $G$ be a connected graph with $N^2(u) \neq \emptyset$, $\forall u \in V(G)$ and let $x, x' \in V(G)$. Then, $d'_G(x,x')$ is defined as the length of the shortest walk $W : x = w_0 \to w_1 \to \ldots \to w_{2k} = x'$ between $x$ and $x'$ of the form $2k\,(k \in \mathbb{N})$ in which $d(w_i, w_{i+2}) = 2,\ i = 0, 2, 4, \ldots, 2k-2$.*

*If there is no such walk between $x$ and $x'$, then $d'_G(x,x') = \infty$.*

**Remark 2.1.**

1. *For any $x, x' \in V(G)$, $d_G(x,x') \leq d'_G(x,x')$.*
2. *For $x, x' \in V(G)$, if $d_G(x,x')$ is even, then $d_G(x,x') = d'_G(x,x')$ and if $d_G(x,x')$ is odd, then $d_G(x,x') < d'_G(x,x')$.*
3. *For a connected bipartite graph $G$, if $x$ and $x'$ are in same partite sets of $G$, then $d_G(x,x')$ is even and hence $d'_G(x,x') = d_G(x,x') < \infty$, whereas, if $x$ and $x'$ are in different partite sets, then $d_G(x,x') < \infty$ but $d'_G(x,x') = \infty$.*

**Example 2.2.** For $G = C_{2n+1}$, if $d_G(x,x')$ is odd, then
$d'_G(x,x') = (2n+1) - d_G(x,x') > d_G(x,x')$.

**Proposition 2.3.** *For a connected, non bipartite and triangle free graph $G$, $d'_G(x,x') < \infty$ for every $x, x' \in V(G)$.*

**Proof.** Let $x, x' \in V(G)$ and $C$ be an odd cycle in $G$. If $d_G(x,x')$ is even, then $d'_G(x,x') = d_G(x,x') < \infty$. Now, if $d_G(x,x')$ is odd, then traversing the cycle $C$, there is an even length walk, say $W$ between $x$ and $x'$. Let
$W : x = u_0 \to u_1 \to u_2 \to \ldots \to u_{2k} = x'$ be the walk. Then, note that, if $d_G(u_i, u_{i+2}) = 1$ for some even $i \leq 2k$, then $u_i, u_{i+1}$ & $u_{i+2}$ forms a triangle in $G$, which is not possible. Hence, $d_G(u_i, u_{i+2}) = 2,\ i = 0, 2, 4, \ldots 2k-2$ and therefore, $d'_G(x,x') \leq 2k < \infty$.

Using this definition of $d'(x,x')$, we obtained the distance formula for two vertices in $G \boxtimes_2 H$.

**Theorem 2.4.** *Let $G$ and $H$ be two graphs with $N^2(x) \neq \emptyset$ for all $x \in V(G) \cup V(H)$. Then for $(x,y), (x',y') \in V(G \boxtimes_2 H)$,*

$$d_{G\boxtimes_2 H}((x,y),(x',y')) = \max\{\tfrac{d'_G(x,x')}{2}, \tfrac{d'_H(y,y')}{2}\}.$$

**Proof.** Suppose $d'_G(x,x') = 2m < \infty$ and $d'_H(y,y') = 2n < \infty$. Consider, the following two walks corresponding to $d'_G(x,x')$ and $d'_H(y,y')$ with $d_G(x_i,x_{i+2}) = 2 = d_H(y_i,y_{i+2})$ $(i = 0,2,4,\ldots)$:

$$P : x = x_0 \to x_1 \to x_2 \to \ldots \to x_{2m-1} \to x_{2m} = x' \ \&$$
$$Q : y = y_0 \to y_1 \to y_2 \to \ldots \to y_{2n-1} \to y_{2n} = y'$$

Without loss of generality, assume that $m \geq n$. Now, $(x,y) \to (x_2,y_2) \to (x_4,y_4) \to \ldots \to (x_{2n},y_{2n}) \to (x_{2n+2},y_{2n}) \to \ldots \to (x',y')$ is a walk in $G \boxtimes_2 H$ of length $m$ as $d_G(x_i,x_{i+2}) = d_H(y_i,y_{i+2}) = 2$. Thus, $d_{G\boxtimes_2 H}((x,y),(x',y')) \leq m = \frac{d'_G(x,x')}{2} = \max\{\frac{d'_G(x,x')}{2}, \frac{d'_H(y,y')}{2}\} < \infty$.

Conversely, let $d_{G\boxtimes_2 H}((x,y),(x',y')) = m < \infty$ and let $P : (x,y) \to (x_1,y_1) \to (x_2,y_2) \to \ldots \to (x_{m-1},y_{m-1}) \to (x',y')$ be corresponding path in $G \boxtimes_2 H$. Now, consider the paths

$$P_G(P) : x = x_0 \to x_1 \to x_2 \to \ldots x_{m-1} \to x_m = x' \ \&$$
$$P_H(P) : y = y_0 \to y_1 \to y_2 \to \ldots \to y_{m-1} \to y_m = y'$$

with $d_G(x_i,x_{i+1}) = 0$ or $2$ for every $i$ and $d_H(y_i,y_{i+1}) = 0$ or $2$ for every $i$.

**Case 1.** $x = x_1 = x_2 = \ldots = x_{m-1} = x_m = x'$

In this case, $d_G(x,x') = 0$ and hence $d'_G(x,x') = 0$ by definition. Also, $y_i \neq y_j$, for any $1 \leq i,j \leq m$, $(i \neq j)$ and hence $d_H(y_{i+1},y_i) = 2$ $(i = 0,2,\ldots)$. Now, as $d_H(y_{i+1},y_i) = 2$ for every $i$, we have $y_i \to a_{i+1} \to y_{i+1}$ for every $i$, for some $a_{i+1} \in V(H)$. Thus, $y = y_0 \to a_1 \to y_1 \to a_2 \to y_2 \to \ldots \to \ldots a_k \to y_m = y'$ with $d_H(y_i,y_{i+1}) = 2$ for every $i$. Hence, we get $d'_H(y,y') \leq 2m$. Therefore, $\frac{d'_H(y,y')}{2} \leq m = d_{G\boxtimes_2 H}((x,y),(x',y'))$. Also, $\frac{d'_G(x,x')}{2} = 0$. Thus, $\max\{\frac{d'_H(y,y')}{2}, \frac{d'_G(x,x')}{2}\} \leq d_{G\boxtimes_2 H}((x,y),(x',y'))$.

Similarly, if $y = y_1 = y_2 = \ldots y_{m-1} = y_m = y'$, then $\max\{\frac{d'_H(y,y')}{2}, \frac{d'_G(x,x')}{2}\} \leq d_{G\boxtimes_2 H}((x,y),(x',y'))$.

**Case 2.** There is atleast one $i$, such that $x_i \neq x$ and $j$ such that $y_j \neq y$.

Note that in this case, from $P_G(P)$ and $P_H(P)$, we get two subsequences $x = u_0 \to u_1 \to u_2 \to \ldots u_{l-1} \to u_l = x'$ with $d_G(u_i,u_{i+1}) = 2$ for every $i$ and $y = v_0 \to v_1 \to v_2 \to \ldots v_{k-1} \to v_k = y'$ with $d_H(v_i,v_{i+1}) = 2$ for every $i$. Now as $d_G(u_i,u_{i+1}) = 2$ and $d_H(v_i,v_{i+1}) = 2$ for every $i$, there is $a_{i+1} \in V(G)$ and $b_{i+1} \in V(H)$ such that $u_i \to a_{i+1} \to u_{i+1}$ and $v_i \to b_{i+1} \to v_{i+1}$. Thus, we get two walks, $x = u_0 \to a_1 \to u_1 \to a_2 \to u_2 \to \ldots a_l \to u_l = x'$ and $y = v_0 \to b_1 \to v_1 \to b_2 \to v_2 \to \ldots \to b_k \to v_k = y'$. Thus $d'_G(x,x') \leq 2l$ and $d'_H(y,y') \leq 2k$.

Therefore, $\frac{d'_G(x,x')}{2} \leq l \leq m$ and $\frac{d'_H(y,y')}{2} \leq k \leq m$. Thus, $\max\{\frac{d'_G(x,x')}{2}, \frac{d'_H(y,y')}{2}\} \leq m$.

Now, suppose $d'_G(x,x') = \infty$ or $d'_H(y,y') = \infty$. If possible, suppose $d_{G\boxtimes_2 H}((x,y),(x',y')) = m < \infty$. Then, using arguments given in converse part, $d'_G(x,x') < \infty$ and $d'_G(y,y') < \infty$, which is a contradiction. Now, suppose $d_{G\boxtimes_2 H}((x,y),(x',y')) = \infty$. If possible, suppose, $d'_G(x,x') < \infty$ and $d'_H(y,y') < \infty$. Then, by arguments given in first part, $d_{G\boxtimes_2 H}((x,y),(x',y')) < \infty$. Thus, $d'_G(x,x') = \infty$ and $d'_H(y,y') = \infty$.

Now, we obtain the connectedness of $G \boxtimes_2 H$.

**Theorem 2.5.** *Let $G$ and $H$ be two connected graphs with $N^2(u) \neq \emptyset$ $\forall\, u \in V(G) \cup V(H)$.*

1. *If $G$ and $H$ are both bipartite graphs, then $G \boxtimes_2 H$ has four components.*
2. *If one of the graph $G$ or $H$ is non-bipartite triangle free and other is bipartite, then $G \boxtimes_2 H$ has 2 components.*
3. *If $G$ and $H$ both are non-bipartite triangle free graphs, then $G \boxtimes_2 H$ is connected.*

**Proof.**

1. Let $U_1$, $U_2$ be two partite sets of $G$ and $V_1$, $V_2$ be two partite sets of $H$ and let $W_{ij} = U_i \times V_j$ for $1 \leq i, j \leq 2$. Now, if $(x,y), (x',y')$ are in different $W_{ij}$, then either $x$ and $x'$ are in different partite sets of $G$ or $y$ and $y'$ are in different partite sets of $H$. Therefore, either $d'_G(x,x') = \infty$ or $d'_H(y,y') = \infty$ and hence, by above theorem, $d_{G\boxtimes_2 H}((x,y),(x',y')) = \infty$. Thus, $G \boxtimes_2 H$ has at least four components $W_{ij}$.

   Now, if $(x,y)$ and $(x',y')$ are in same $W_{ij}$, $(i,j \in \{1,2\})$, then $x$ & $x'$ are in same partite set of $G$ and $y$ & $y'$ are in same partite set of $H$. Thus, $d'_G(x,x') < \infty$ and $d'_H(y,y') < \infty$ and hence, $d_{G\boxtimes_2 H}((x,y),(x',y')) < \infty$, i.e., $(x,y)$ and $(x',y')$ are connected by a path in $G \boxtimes_2 H$.
   Therefore, there are exactly four components in $G \boxtimes_2 H$.

2. Let $G$ be a bipartite graph and let $U_1$ and $U_2$ be two partite sets of $G$ and let $W_i = U_i \times V(H)$ for $1 \leq i \leq 2$. If $(x,y)$ and $(x',y')$ are in different $W_i$, then $x$ and $x'$ are in different partite sets of $G$ and hence, $d'_G(x,x') = \infty$. Therefore, $d_{G\boxtimes_2 H}((x,y),(x',y')) = \infty$ and so, there are at the least two components in $G \boxtimes_2 H$.

Further, if $(x,y)$ and $(x',y')$ are in same $W_i$, $1 \leq i \leq 2$, then $x$ and $x'$ are in same partite set and hence, $d'_G(x,x') < \infty$. Also, as $H$ is non bipartite and triangle free, by Proposition 2.3, $d'_H(y,y') < \infty$. Thus, $d_{G\boxtimes_2 H}((x,y),(x',y')) < \infty$ and hence $(x,y)$ and $(x',y')$ are connected by a path. Therefore, there are two components in $G \boxtimes_2 H$.

3. Let $(x,y),(x',y') \in V(G \boxtimes_2 H)$. Then, by Proposition 2.3, $d'_G(x,x') < \infty$ and $d'_H(y,y') < \infty$. Thus, $d_{G\boxtimes_2 H}((x,y),(x',y')) < \infty$ and hence, $(x,y)$ and $(x',y')$ are connected by a path. Thus, $G \boxtimes_2 H$ is connected.

## 3. Diameter, Eccentricity and Radius

In this section, we obtain some more basic graph parameters which are depending on distance between two vertices. For example, diameter, eccentricity and radius.

**Definition 2.1.** *Let $G = (V(G), E(G))$ be a graph. Then, we define*

1. *2-diameter of $G$ by $\max\{d'_G(x,y) : x,y \in V(G)\}$ and we denote it by $diam_2(G)$.*
2. *2-eccentricity of a vertex $x$ in $G$ as $\max\{d'_G(x,x') : x' \in V(G)\}$ and we denote it by $e'(x)$.*
3. *2-radius of $G$ as $\min\{e'(x) : x \in V(G)\}$ and we denote it by $rad_2(G)$.*
4. *2-closed neighborhood of a vertex $x$ in $G$ as $\{x' \in V(G) : d'_G(x,x') \leq i\}$ and we denote it by $N'_i[x,G]$.*

If we replace $d'_G(x,x')$ by $d_G(x,x')$ in above definitions, we get the definition of usual diameter. eccentricity, radius and closed neighborhood.

**Remark 3.2.** *In general, $diam(G)$ (or $rad(G)$) and $diam_2(G)$ (or $rad_2(G)$) are different. Note that, $diam_2(G)$ (or $rad_2(G)$) is always even and if $diam(G)$ (or $rad(G)$) is even, still $diam_2(G)$ (or $rad_2(G)$) may be different from $diam(G)$ (or $rad(G)$).*

We obtained the following relation between diameter of graph $G \boxtimes_2 H$ and 2-diameter of $G$ and $H$.

**Theorem 3.3.** *Let $G$ and $H$ be two graphs. Then,*

$$diam(G \boxtimes_2 H) = \max\{\tfrac{diam_2(G)}{2}, \tfrac{diam_2(H)}{2}\}.$$

**Proof.** Let $diam(G \boxtimes_2 H) = d$. Then, for any $x,x' \in V(G)$ and $y,y' \in V(H)$, $d_{G\boxtimes_2 H}((x,y),(x',y')) \leq d$. Therefore, by Theorem 2.4, $\max\{\frac{d'_G(x,x')}{2}, \frac{d'_H(y,y')}{2}\} \leq d$.

Thus, $d'_G(x,x') \le 2d$ & $d'_H(y,y') \le 2d$. But, as $x,x'$ and $y,y'$ are arbitrary, $diam_2(G) \le 2d$ and $diam_2(H) \le 2d$. Therefore, $\max\{\frac{diam_2(G)}{2}, \frac{diam_2(H)}{2}\} \le d$.

Conversely, let $diam_2(G) = d_1$ and $diam_2(H) = d_2$. Then, for any $(x,y),(x',y') \in V(G \boxtimes_2 H)$, $d'_G(x,x') \le d_1$ and $d'_H(y,y') \le d_2$. Therefore, $d_{G\boxtimes_2 H}((x,y),(x',y')) = \max\{\frac{d'_G(x,x')}{2}, \frac{d'_H(y,y')}{2}\} \le \max\{\frac{d_1}{2}, \frac{d_2}{2}\}$. But, as $(x,y)$ and $(x',y')$ are arbitrary, $diam(G \boxtimes_2 H) \le \max\{\frac{d_1}{2}, \frac{d_2}{2}\}$.

By similar arguments, the following can be proved.

**Theorem 3.4.** *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs.*

1. *For $(x,y) \in V(G \boxtimes_2 H)$, $ecc(x,y) = \max\{\frac{e'(x)}{2}, \frac{e'(y)}{2}\}$.*
2. *$rad(G \boxtimes_2 H) = \max\{\frac{rad_2(G)}{2}, \frac{rad_2(H)}{2}\}$.*
3. *For $(x,y) \in V(G \boxtimes_2 H)$, $N_i[(x,y), G \boxtimes_2 H] = N'_{2i}[x,G] \times N'_{2i}[y,H]$, where $N_i[x,G] = \{x' \in V(G) : d(x,x') \le i\}$.*

We consider the following non-bipartite graphs:

**Definitions** [1]

1. A wheel graph $W_n$ is a graph with $n+1$ vertices that contains $C_n$ and one other vertex which is adjacent to every vertex of $C_n$. The vertex which is adjacent to every vertex is called center vertex.
2. The Helm graph $H_n$ is the graph obtained from $W_n$ by adding a pendent edge to each vertex of $C_n$ in $W_n$.
3. The closed Helm graph $CH_n$ is the graph obtained from $H_n$ by adding edges between pendent vertices.

**Example 3.5.**

| **Graph** | $diam(G)$ | $diam_2(G)$ | $rad(G)$ | $rad_2(G)$ |
|---|---|---|---|---|
| $C_{2n+1}$ $(n > 1)$ | $n$ | $2n$ | $n$ | $2n$ |
| $H_n$ $(n \ge 4)$ | 4 | 6 | 2 | 4 |
| $CH_n$ $(n \ge 4)$ | 4 | 6 | 2 | 4 |
| $P_n$ $(n > 2)$ | $n-1$ | $\infty$ | $[n]$ | $\infty$ |
| $K_{m,n}$ $(m,n > 2, m \ne n)$ | 2 | $\infty$ | 2 | $\infty$ |
| $W_n$ $(n > 4)$ | 2 | $\infty$ | 2 | $\infty$ |

## References

[1] Acharya U. P., A study of new generalized product of graphs, Thesis, Sardar Patel University, 2016, India.

[2] Acharya U. P. and Mehta H. S., 2-Cartesian product of special graphs, Int. J. of Math and Soft Computing, 4(1), (2014), 139-144.

[3] Acharya U. P. and Mehta H. S., 2-Tensor product of graphs, Int. J. of Math and Scientific Computing, 4(1), (2014), 21-24.

[4] Balakrishnan R. and Ranganathan K., A textbook of graph theory, Springer, New York, 2000.

[5] Hammack R., Imrich W. and Klavzar S., Handbook of product graphs, CRC Press, New York 2011.

[6] Imrich W. and Klavzar S., Product graphs: Structure & recognition, Wiley, New York, 2000.

[7] Mehta H. S. and Acharya U. P., Degree & distance in 2-Cartesian product of graphs, Research & reviews: Discrete Mathematical structures, 5(3), (2018), 11-14.